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AN ANALYSIS OF SYSTEMATIC AND RANDOM ERRORS IN THE SWEDISH MOTORIZED PRECISE LEVELLING TECHNIQUE

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0. Abstract

The Swedish motorized levelling technique was introduced in 1974. The method has been continuously improved and today it meets the requirements of precise levelling.

The 1979 field season data was used for the analysis. It consists of 62 lines combined to 21 loops. The RMS loop closing errors are 0.8 mm/$\sqrt{km}$ and 2.1 mm/$\sqrt{km}$ for two-ways and one-way measured lines, respectively. No systematic errors were detected in the two-ways measured lines. However, the large difference between one-way and two-ways RMS errors indicates systematic and/or correlated random errors in one-way measurements. Possible systematic errors are obviously eliminated in the means from forward and backward observations. A strong negative correlation (-0.7) was found between the one-way levellings.

An error model was introduced including a systematic error in one-way levelling (but not in the two-ways means). Moreover the model includes correlated random errors. The model is consistent with the variances obtained from loop misclosures and from the forward-backward observation differences of the lines. The estimated systematic error is of magnitude 0.24 mm/$\sqrt{km}$ and the standard error of one-way levelling is 1.0 mm/$\sqrt{km}$. We found a correlation coefficient 0.1 among random errors forward-backward, and a correlation coefficient 0.4 between neighbouring sections.

1. Background

In 1974 a motorized group consisting of two surveyors and two assistants, an instrument vehicle and two vehicles with levelling rods, started levelling in Sweden. Originally the unit was used for second order levelling. The experiences gained since the initial stage have been used continuously to improve all technical details of the system as well as practical routines.

Today the automated technique meets first order requirements and subsequently the method is being used in the third remeasurement of the Swedish first order levelling network under way. During the field season 1981 five measuring parties are planned being operating. Some particular advantages of the method are listed below (from Becker, 1977):

1.1 The negative effects of atmospheric refraction are reduced due to the increase in the height of the instrument above the ground: 2.0 m compared with an average height of 1.5 m for other techniques. According to Kukkamäkis well known formula the effect of refraction is, thereby, reduced by a half.

1.2 Errors due to uneven loading which can arise when an observer moves around the instrument or when the assistant moves around the staff are reduced.

1.3 Due to the speed of the observations, errors due to settlement or rising of the instrument and rods are reduced.

1.4 More stable rods supports and rods base-plates facilitate more accurate readings on the rods.

1.5 Continuous checks of the consistency of observations and closures are possible in the field, thanks to the use of an electric calculator with storage facilities and print-out.

1.6 Better quality observations are possible due to the use of two observers working under relatively short periods.

1.7 Rapid transport permits greater flexibility. The work is physically less tiring and, due to wind-shields etc, can be carried out under relatively poor weather conditions.

1.8 Production is increased and costs decreased.

On the other hand one can expect the following disadvantages of the automated method (from Becker, ibid.)

2.1 The size of the vehicles imposes restrictions on the location of lines along narrow roads. The road must be wide enough for the cars to pass each other.

2.2 The location of bench-marks, which must be so placed as to minimize the amount of foot levelling, can give rise to problems.

2.3 Traffic intensity can limit the choice of set-ups.

2.4 Training costs are initially high.

2.5 Refraction effects due to heat radiation from the vehicles can arise under certain conditions. (This has not, however, been a problem in second-order levelling.)

2.6 Observing routines can give rise to errors of a systematic character.

2.7 Initially high investment costs.

For more details on the motorized levelling technique we refer to Becker (1977 and 1980).

It is our intention to investigate the 1979 field season data with respect to possible systematic and random effects.

2. Systematic Versus Random Errors

The study of systematic and accidental (random) errors in precise levelling has been of great interest among geodesists over periods of decades. Besides the traditional use of precise levelling for the establishment of national first order vertical networks it has gained an increasing interest as one of the most accurate methods for studying geodynamics and engineering deformations.

The occurrence of both systematic and accidental errors in the observations and their detection and separation have been and are still challenging problems. One apparently simple way to overcome the difficulties in discriminating between the systematic and accidental origin of errors would be to follow the recommendations by IAG 1948 (Braaten et.al., 1950), assuming that systematic errors become random after a certain distance. However such a strategy has to be rejected from theorectical point of view. In contrast to a systematic error a random error vanishes when the measurement is repeated to infinity (expectation).

Lucht (1972) pointed out that the large differences of various types of variance estimates do not necessarily indicate systematic errors but can be completely explained by the precence of correlation among random errors. Then the question is still open whether the main expected error source is of systematic nature or not. If the answer can be shown to be in the affirmative there is hope that one or several essential error sources can be detected and eliminated in the measuring procedure.

Naturally, this kind of analysis becomes of particular interest when introducing a new measuring technique such as the Swedish motorized levelling.

3. The Data

Our test data were taken from the 1979 field season production of automated levelling (with some complementary observations in May 1980). It consists of height difference observations (forward and backward) and distances between all fixes of the area. These data were merged for 62 lines consisting of 1481 sections of approximately 1 km of length. The lines were combined into 21 loops (polygons). Table 1 includes the following parameters of the loops:

Parameter No.	Characteristic
1	number of sections
2	sum of squares of mean elevation differences of sections ($\bar{h}^2$)
3	sum of products distance (d) times $\bar{h}$
4	sum of products d^2 times $\bar{h}$
5	one-way closing error (forward)
6	- " - (backward)
7	two-ways closing error

The observations are not corrected for refraction (cf Kukkamäki, 1938). Furthermore they are not corrected to potential differences. (The systematic error introduced by neglecting the nonparallelism of the level surfaces is assumed of no relevance in the relatively small area.)

Table 1. Charateristics of the 21 loops. The parameters mean the algebraic sums with negative contributions from lines running in opposite direction to loops.

Parameter No.			1	2	3	4	5	6	7
Loop No.	Total No.of Sect. (n)	Total Dist. D [Km]	No.of [m]	$\Sigma \bar{h}^2$ [$m^2 x 10^4$]	$\Sigma d x \bar{h}$ [mxkm]	$\Sigma d^2 x \bar{h}$ [$mxkm^2$]	Σh_f [mm]	Σh_f [mm]	w [mm]
1	129	132.3	99	1.641	-13.05	- 23.48	6.75	3.52	5.12
2	124	111.6	42	0.286	-40.82	- 61.53	32.41	-19.60	6.37
3	101	93.13	-7	1.020	19.80	23.46	18.08	8.25	13.15
4	99	102.0	-1	0.419	36.09	66.82	-20.00	29.18	4.56
5	101	106.0	61	0.534	0.64	- 2.00	34.50	-27.54	3.49
6	110	115.1	20	0.768	70.01	131.30	-17.53	8.09	-4.70
7	97	104.5	-65	-2.054	-32.92	- 40.55	-34.07	30.88	-1.61
8	104	109.0	-26	-0.944	-54.19	-129.1	20.51	-17.93	1.34
9	128	113.4	128	2.112	12.99	25.14	9.47	- 5.90	1.69
10	113	120.6	-19	-0.202	-14.08	- 28.42	- 7.78	- 0.66	-4.20
11	103	111.4	73	1.014	20.41	46.07	10.86	-45.38	-12.76
12	124	117.2	-22	-0.586	-14.79	- 19.50	53.91	22.29	15.81
13	109	101.4	45	0.709	- 3.491	- 12.63	-11.42	-12.65	-12.07
14	148	138.8	76	0.995	-28.22	- 48.50	4.45	14.42	9.41
15	145	130.9	75	-0.427	7.08	6.00	- 0.25	-11.22	- 5.75
16	120	115.0	120	1.095	13.38	18.81	7.33	- 1.14	3.01
17	111	104.9	95	0.579	- 0.72	- 2.41	-10.99	36.95	12.93
18	113	104.2	-83	-0.236	3.56	12.43	- 3.41	13.36	4.98
19	120	121.0	36	0.135	3.53	7.27	- 8.96	-16.11	-12.54
20	87	86.62	63	0.202	8.62	14.04	41.57	-21.86	9.84
21	109	104.9	-25	-0.196	24.87	51.67	- 0.17	- 0.15	- 0.13

n = number of sections (between bench marks)

w = two-ways closing error

h_f, h_b = forward, backward height difference between bench-marks

d = section distance

$\bar{h} = (h_f + h_b)/2$

4. Analysis of Systematic Errors

If systematic errors were accumulating in the levelling procedure there would be a possibility to detect these from the observed closing errors (w). For example, if each setup of the instrument causes a constant error (bias) there would be a correlation between the number of bench-marks of the loop (parameter 1) and the closing error. The same technique was used by Remmer (1980) for the analysis of the systematic refraction error (mainly proportionalto parameter 7). We will make this study for all seven parameters as listed in Table 1.

4.1 Correlation Coefficients

First we compute the correlation coefficient between each parameter (x) and w in the following way

$$r_{xw} = \frac{\sum_{i=1}^{21} \left(x_i-\bar{x}\right)\left(w_i-\bar{w}\right)}{\left\{\sum_{i=1}^{21} \left(x_i-\bar{x}\right)^2 \sum_{i=1}^{21} \left(w_i-\bar{w}\right)^2\right\}^{1/2}} \qquad (1)$$

where $\bar{x}$ and $\bar{w}$ are the mean values of x and w, respectively. (In fact each parameter (including w) were normalized to parameter/$\sqrt{\text{kilometer}}$ in the determination of r_{xw}). The results are shown in Table 2.

Table 2. Correlation coefficients of normalized parameters versus w

Parameter	1	2	3	4	5	6
r_{xw}	-0.023	0.057	-0.153	-0.158	0.453	0.313
$T=\left\lvert\frac{r_{xw}\sqrt{19}}{\sqrt{1-r_{xw}^2}}\right\rvert$	0.100	0.249	0.674	0.697	2.215	1.437

In the table is given also a test parameter T, which, in the case of a significant correlation, should exceed

t (19) = 1.73 (1.33) at the risk level 5% (10%).

See Hald (1952, p. 609). At the risk level 5% only the one-way forward closing error is significantly correlated with w. At the 10% level both one-way correlations are significant.

Next we repeat the application of formula (1) with w substituted by each of the one-way closing errors (parameters 5 and 6). The results are shown in Tables 3 and 4.

Table 3. Correlation coefficients of normalized parameters versus $y = \Sigma h_f$

Parameter	1	2	3	4	6	7
r_{xy}	0.208	0.159	-0.217	-0.245	-0.705	0.453
$T = \left\| \frac{r_{xy}\sqrt{19}}{\sqrt{1-r^2_{xy}}} \right\|$	0.927	0.702	0.969	1.102	4.333	2.215

Table 4. Correlation coefficients of normalized parameters versus $y = \Sigma h_b$

Parameter	1	2	3	4	5	7
r_{xy}	-0.238	-0.122	0.109	0.135	-0.705	0.313
$T = \left\| \frac{r_{xy}\sqrt{19}}{\sqrt{1-r^2_{xy}}} \right\|$	1.068	0.536	0.478	0.594	4.333	1.437

We conclude that there is a strong negative correlation (-0.7) between the one-way levellings. There is a weak correlation between one-way levellings and two-ways (significant at the risk levels 5% and 10% for forward and backward levelling, respectively). Otherwise no significant correlations are detected.

We conclude that there is no significant correlation between the studied parameters and the closing errors.

4.2 Regression Analysis

We now make a regression analysis of each of the parameters. Suppose that the levelling procedure suffers from an incremental systematic error $c\,\Delta x_\ell$ for the observation between two bench marks. Here c is an (unknown) constant and Δx_ℓ is the contribution to the parameter x ($x = \sum_{loop} \Delta x_\ell$).

The constant c is determined by the least squares method such that the weighted sum of squares of

$$w_i + cx_i$$

is a minimum, i.e.

$$\sum_{i=1}^{21} \frac{1}{D_i} (w_i + cx_i)^2 = \text{minimum}$$

where D_i is the distances. The solution is

$$\hat{c} = \frac{-\sum_i w_i x_i / D_i}{\sum_i x_i^2 / D_i} \qquad (2)$$

and the estimated variance becomes

$$s_c^2 = \frac{s^2}{\sum_i x_i^2 / D_i} \qquad (3)$$

where

$$s^2 = \frac{1}{20} \sum_{i=1}^{21} \frac{1}{D_i} (w_i + \hat{c} x_i)^2 \qquad (4)$$

The computed correction factors ($\hat{c}$) are listed in Table 5 along with there error estimates. Finally we illustrate in Table 6 the reduced RMS errors (errors after correction).

$$s\sqrt{20/21} \qquad (5)$$

Table 5. Correction factors ($\hat{c}$) and error estimates (s_c).

Parameter	1	2	3	4	5	6
Units	mm/section	mm/m^2	mm/(mxkm)	$mm/(mxkm^2)$	dim.less	dim.less
$\hat{c}$	-0.011	-1.14×10^{-4}	+0.044	+0.024	-0.179	-0.110
s_c	0.028	1.96×10^{-4}	0.069	0.036	0.072	0.087

Table 6. Reduced RMS Errors {formula (5)}. Units: $mm/\sqrt{km}$

Parameter	1	2	3	4	5	6
RMS Error	0.796	0.793	0.792	0.791	0.698	0.769

Table 5 shows that the error estimates (s_c) are of the same order of magnitude as the estimated parameters 1-4. Comparing Table 6 and the RMS loop discrepancy 0.80 mm/$\sqrt{km}$ as computed from Table 1, we notice that there is very little to gain in correcting for any of the parmeters 1-4. Thus we conclude that none of these parameters differs significantly from zero.

The test parameter 5 (Σh_f) is significant at the 5% risk level (T=0.179/0.072=2.49>t(20)=1.73). Thus the mean closing errors may be improved by a correction $-0.18 \times \Sigma h_f$.

A repetition of the above computations by formulas (2)-(5) with w substituted by Σh_f and Σh_b, respectively, did not reveal any significant parameters of types 1-4 related to one-way levelling. Thus we have not found the origin of the strong correlation between the one-way levellings.

5. Comparison of Loop Errors

In Table 7 we compare the loop errors $(w)/\sqrt{D}$ and the one-way observation error differences $(\Sigma h_f-\Sigma h_b)/\sqrt{D}$ for the 21 loops. If there were no systematic errors the expected value of any of these quantities would be zero. Then we may compute the following independent estimates of the variance of the levelling (two-ways means)

$$s_w^2 = \frac{1}{21}\sum_{i=1}^{21} w_i^2/D_i \qquad (6)$$

$$s_{\Delta h/2}^2 = \frac{1}{21 \times 4}\sum_{i=1}^{21}\left\{\Sigma(h_f-h_b)\right\}^2/D_i, \qquad (7)$$

where D_i is the length of loop i. The result is

$$s_w^2 = 0.640 \quad mm^2/km$$

and

$$s_{\Delta h/2}^2 = 3.679\ mm^2/km$$

Let us form the test parameter

$$T = \frac{s_{\Delta h/2}^2}{s_w^2} = \frac{3.679}{0.640} = 5.749$$

We can now test the new hypothesis that the two variances are identical:

$$H_0 : \sigma_{\Delta h/2}^2 = \sigma_w^2$$

However, as

$$T > F_{0,99}(21,21) = 2.86,$$

where F is taken from an F-distribution table, we reject H_0 and conclude that the two variances are not equal at the

Table 7. Comparison of loop errors from two-ways (w) and one-way closing errors (Σh_f and Σh_b). Units: mm/$\sqrt{km}$.

Lop No	$w/\sqrt{D}$	$\Sigma(h_f - h_b)/\sqrt{D}$	$\Sigma h_f/\sqrt{D}$	$\Sigma h_b / \sqrt{D}$
1	0.445	0.281	0.587	0.306
2	0.603	4.923	3.068	-1.855
3	1.363	1.019	1.873	0.855
4	0.451	-4.870	-1.980	2.889
5	0.339	6.026	3.351	-2.675
6	-0.438	-2.388	-1.634	0.754
7	-0.158	-6.354	-3.333	3.021
8	0.129	3.682	1.965	-1.717
9	0.159	1.443	0.889	-0.554
10	-0.383	-0.648	-0.708	0.060
11	-1.209	6.182	1.882	-4.300
12	1.460	7.039	4.980	-2.059
13	-1.198	0.122	-1.134	-1.256
14	0.819	-0.868	0.388	1.256
15	-0.503	0.959	-0.022	-0.981
16	0.281	0.790	0.684	-0.106
17	1.262	4.681	-1.073	3.698
18	0.488	1.643	-0.334	1.309
19	-1.140	0.650	0.815	-1.465
20	1.058	6.815	4.467	-2.349
21	-0.013	-0.002	-0.017	-0.015
Mean square	0.6400	14.717	4.735	3.905
RMS	0.800	3.836	2.176	1.976

1 % risk level. The difference is due to systematic errors and/or correlation among observations {cf. formulas (14) and (15)}. In the same way we can test whether the variances as computed from the one-way closing errors are identical. Then we obtain the test parameter (cf. Table 7)

$$T = \frac{s_f^2}{s_b^2} = \frac{4.735}{3.905} = 1.213 ,$$

from which we conclude that

$$\sigma_f^2 = \sigma_b^2$$

at the risk level 1%. Thus we can form the following pooled variance for one-way measurements

$$s_p^2 = \frac{s_f^2+s_b^2}{2} = 4.32$$

which implies

$$s_p = 2.08 \text{ mm}/\sqrt{\text{km}}$$

This result indicate that there is a considerable systematic effect and/or correlation in the one-way observations. If the error is of a systematic nature, it is (partly or completely) eliminated in the mean value computations from forward and backward measurements. In section 4 we did not detect any correlation with any of the parameters 1-4. In the next chapter we give an error model consistent with the above variances.

6. Error Model

We now give a possible error model for the levelling. Let $a\sqrt{d}$ denote the systematic error in forward measurement between two bench-marks. Then the systematic error from the backward measurements becomes $- a\sqrt{d}$ between two bench marks. a is the error in units of $\text{mm}/\sqrt{\text{km}}$ and d is the average bench-mark distance. Let the total distance of the i-th loop be denoted D_i

$$D_i = n_i \, d, \qquad (8)$$

where n_i is the number of sections of the loop. Furthermore, let the total random error of the forward and backward observations of the i-th loop be denoted $(\tilde{\varepsilon}_f)_i$ and $(\tilde{\varepsilon}_b)_i$ respectively, i.e.

$$(\tilde{\varepsilon}_f)_i = \sum_{k=1}^{n_i} (\varepsilon_f)_k \sqrt{d} \qquad (9a)$$

$$(\tilde{\varepsilon}_b)_i = \sum_{k=1}^{n_i} (\varepsilon_b)_k \sqrt{d} \qquad (9b)$$

where $(\varepsilon_f)_n$ and $(\varepsilon_b)_n$ are the random errors (forward and backward) per (bench-mark distance)$^{\frac{1}{2}}$ with properties

$$E\{(\varepsilon_f)_k^2\} = \sigma^2 \qquad (10a)$$

$$E\{(\varepsilon_f)_k(\varepsilon_b)_k\}=\sigma^2\rho_I \qquad (10b)$$

where ρ_I is the correlation coefficient between forward and backward observations. Moreover we assume that observations from neighbouring bench-mark distances are correlated with correlation coefficient ρ_{II}, i.e.

$$E\{(\varepsilon_f)_k(\varepsilon_f)_{k+1}\}=\sigma^2\rho_{II} \qquad (10c)$$

Now we are ready to express the expectations of the previously defined variances s_w^2, $s^2_{\Delta h/2}$ and s_p^2 in terms of a^2, σ^2, ρ_I and ρ_{II}. From (8) - (10) we obtain for the i-th loop

$$\frac{1}{D_i}E\{(\sum_{k=1}^{n_i} h_{f_k})^2\} = \frac{d}{D_i}\ E\{\Big(\sum_{k=1}^{n_i}(a+(\varepsilon_f)_k)\Big)^2\} \approx$$

$$\approx \frac{m_i^2}{n_i}a^2 + \frac{1}{n_i}\Sigma_k\Sigma_\ell\ E\{(\varepsilon_f)_k(\varepsilon_f)_\ell\}=\frac{m_i^2a^2}{n_i} + \sigma^2(1+2\rho_{II}) \qquad (11)$$

where m_i is the algebraic sum of number of sections (with negative contributions from lines running opposite to loop The same expected value is obtained if we replace h_f by h_b.

Furthermore

$$\frac{1}{D_i}E\{\Big(\sum_{k=1}^{n_i}(h_{f_k}-h_{b_k})\Big)^2\} = 4a^2\frac{m_i^2}{n_i} + 2\sigma^2(1+2\rho_{II})(1-\rho_I) \qquad (12)$$

Hence,

$$E\{s_p^2\}=a^2k+\sigma^2\ (1+2\rho_{II}) \qquad (13)$$

$$E\{s^2_{\Delta h/2}\} = a^2k + \frac{1}{2}\sigma^2(1+2\rho_{II})(1-\rho_I) \qquad (14)$$

$$E\{s_w^2\} = \frac{1}{2}\sigma^2(1+2\rho_{II})(1+\rho_I) \qquad (15)$$

where (from Table 1)

$$k = \frac{1}{21}\sum_{i=1}^{21} m_i^2/n_i = 37.566$$

It is easily shown that the variances s_p^2, s_w^2 and $s_{\Delta h/2}^2$ are related by

$$s_p^2 = s_w^2 + s_{\Delta h/2}^2$$

Hence we have only two independent equations with four unknowns (σ^2, a^2, ρ_I, ρ_{II}). Subsequently we need two more independent equations for a successful solution.

7. Analysis of Levelling Lines

In this chapter we are going to analyse the errors of all 62 lines included in the network.

Let Δ_i denote the discrepancy between forward and backward height difference measurements $(h_f)_i$ and $(h_b)_i$ between two bench-marks, i.e.

$$\Delta_i = \{(h_f)_i - (h_b)_i\}/\sqrt{d_i}$$

In accordance with formula (12) we get

$$E\{\Delta_i^2\} = 4a^2 + 2\sigma^2(1-\rho_I) \qquad (16)$$

where a is the systematic error and ρ_I is the correlation coefficient between $(\varepsilon_f)_i$ and $(\varepsilon_b)_i$

From each line we compute the variance

$$s_I^2 = \frac{1}{2n} \sum_{i=1}^{n} \Delta_i^2$$

where n is the number of bench-mark distances of the line.

Formula (16) yields

$$\sigma_I^2 = E\{s_I^2\} = 2a^2 + \sigma^2(1-\rho_I) \qquad (17)$$

We now study the product of two neighbouring differences Δ_i and Δ_{i+1}, obtaining

$$E\{\Delta_i \Delta_{i+1}\} = E\{\left(2a + (\varepsilon_f)_i - (\varepsilon_b)_i\right)\left(2a + (\varepsilon_f)_{i+1} - (\varepsilon_b)_{i+1}\right)\} =$$

$$= 4a^2 + 2\sigma^2(1-\rho_I)\rho_{II}$$

where

$$\rho_{II} = \frac{1}{\sigma^2} E\{(\varepsilon_f)_i(\varepsilon_f)_{i+1}\} = \frac{1}{\sigma^2} E\{(\varepsilon_b)_i(\varepsilon_b)_{i+1}\}$$

Hence the expected value of the variance

$$s^2_{III} = \frac{1}{4(n-1)} \sum_{i=1}^{n-1} (\Delta_i - \Delta_{i+1})^2$$

becomes

$$\sigma^2_{III} = E\{s^2_{III}\} = \frac{1}{4(n-1)} \sum_{i=1}^{n-1} E\{\Delta_i^2 + \Delta_{i+1}^2 - 2\Delta_i\Delta_{i+1}\} =$$

$$= 2a^2 + \sigma^2(1-\rho_I) - \frac{2}{4}\{4a^2 + 2\sigma^2(1-\rho_I)\rho_{II}\} =$$

$$= \sigma^2(1-\rho_I)(1-\rho_{II}) \qquad (18)$$

Moreover we compute the covariance

$$s_{IV} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \Delta_i\Delta_{i+1}$$

with the expected value

$$\sigma_{IV} = E\{s_{IV}\} = 2a^2 + \sigma^2(1-\rho_I)\rho_{II} \qquad (19)$$

From (17) - (19) we obtain the following relation between the estimated variance components

$$s_I^2 - s_{IV} = s^2_{III} \qquad (20)$$

Again we notice that only two of these equations are independent.

7.1 Analysis for Purely Random Errors

In this section we simplify the problem by assuming that there is no systematic error i.e.

$$a = 0$$

Then (17) - (19) yield the following estimates of ρ_{II}:

$$r_{II} = s_{IV}/s_I^2 \qquad (21a)$$

or

$$r_{II} = (s_I^2 - s_{III}^2)/s_I^2 \qquad (21b)$$

In Table 8 we give s_I^2, s_{III}^2, s_{IV} and r_{II} as computed from (19a) and (19b) for all lines included in the 21 loops.

The pooled results from all 62 lines are

$$s_I = 0.814 \text{ mm}/\sqrt{\text{km}}$$

$$s_{III} = 0.584 \text{ mm}/\sqrt{\text{km}}$$

$$r_{II} = 0.42 \pm 0.04$$

where the standard deviation of r_{II} was computed from

$$s_{r_{II}}^2 = \frac{\sum_{k=1}^{62} p_k (r_{II}^{(k)})^2 - r_{II}^2 \Sigma p_k}{61 x \sum_k p_k}$$

where

$$p_k = n_k - 1$$

$$r_{II}^{(k)} = \text{estimate from line k}$$

The number of redundancies equals $\sum_k p_k - 1 = 1418$

Hence at the risk level 1% we get the following

confidence interval for ρ_{II}

$$0.32 \leq \rho_{II} \leq 0.52 \quad (a \equiv 0)$$

This correlation coefficient is valid between one-way levelled lines. For the mean of forward-backward levelling we get accordingly the correlation coefficient $\bar{\rho}_{II}$:

$$\bar{\rho}_{II} = \frac{1}{4\sigma^2} E\{\left((\varepsilon_f)_i + (\varepsilon_b)_i\right)\left((\varepsilon_f)_{i+1} + (\varepsilon_b)_{i+1}\right)\} = \frac{\rho_{II}}{2}(1+\rho_I) \qquad (22)$$

For the estimation of $\bar{\rho}_{II}$ we need an estimate of ρ_I. In case a = 0 we obtain from (14) and (15).

$$\frac{s_{\Delta h/2}^2}{s_w^2} = \frac{1-r_I}{1+r_I}$$

which implies

$$r_I = \frac{s_w^2 - s_{\Delta h/2}^2}{s_w^2 + s_{\Delta h/2}^2}$$

Inserting

$$s_w^2 = 0.640 \quad \text{and} \quad s_{\Delta h/2}^2 = 3.684$$

we get the solutions

$$r_I = -0.7040$$

$$\bar{r}_{II} = 0.062$$

and finally from (17)

$$s^2 = s_I^2/(1-r_I) = 0.3889 \text{ mm}^2/\text{km} \quad (f = 1419)$$

However, due to the a priori assumption that $a \equiv 0$ this solution is not compatible with the variances of chapters 6-7 {formulas (13)-(15), (17)-(19)}. For example, we get another estimate of s^2 by (15):

$$s^2 = \frac{2s_w^2}{(1+2r_{II}(1+r_I)} = 2.350 \text{ mm}^2/\text{km} \quad (f = 18)$$

Using the test parameter

$$T = \frac{2.350}{0.3889} = 6.04 > F(18,1419) = 1.93$$

we conclude that the two estimates of s^2 are significantly different at the 1% risk-level. Subsequently the hypothesis $a = 0$ has to be rejected.

7.2 Estimation of the One-Ways Differences

In Table 8 we have included also a column with

$$\sum_{i=1}^{n} \Delta_i/n$$

Table 8. Variances s_I^2, s_{III}^2, covariance s_{IV} and correlation coeficient r_{II} between neighbouring sections in case a=0.
Units for variance components: mm^2/km. n= number of sections.
$\Delta_i=(h_f)_i-(h_b)_i$ $(mm/\sqrt{km})$.

Line	n	$\sum_{i=1}^{n}\Delta_i/n$	s_I^2	s_{III}^2	s_{IV}	r_{II}(19a)	r_{II}(19b)
3413	51	-0.3784	0.5430	0.2720	0.2746	0.499	0.506
3501	42	0.7920	0.7538	0.3116	0.4360	0.587	0.578
3504	27	1.2443	1.7455	0.3860	1.3626	0.779	0.781
3509	57	0.8457	1.2817	0.4940	0.7902	0.615	0.616
3611	9	0.3105	0.3580	0.2744	0.0172	0.234	0.048
3614	16	0.1751	0.2883	0.1193	0.1787	0.586	0.620
1330	27	0.1799	0.3160	0.2837	0.0299	0.102	0.095
1331	2	2.6398	3.6648	0.3612	3.3036	0.901	0.901
1332	8	-0.7270	0.7587	0.2271	0.4762	0.701	0.628
1333	41	0.7465	0.9590	0.4454	0.5099	0.536	0.532
1334	13	0.5576	0.8067	0.3288	0.4170	0.592	0.517
1335	20	0.1368	0.5085	0.3176	0.1717	0.375	0.338
1336	12	-0.4594	0.4711	0.3258	0.1690	0.308	0.359
1337	13	-0.1916	0.2680	0.4304	-0.1402	-0.606	-0.523
1338	24	0.5353	0.7169	0.4592	0.2597	0.359	0.362
1339	31	-0.1143	0.4989	0.3811	0.1315	0.236	0.264
1340	36	0.2772	0.8227	0.4067	0.3694	0.506	0.449
1341	15	-0.4780	0.4212	0.3552	0.0889	0.157	0.211
1342	14	0.0205	0.3883	0.4858	-0.1493	-0.251	-0.384
1343	39	-0.9449	0.8280	0.1929	0.6315	0.767	0.763
1344	16	1.2801	1.4081	0.6582	0.8078	0.533	0.574
1345	26	1.0025	0.8159	0.2639	0.5426	0.672	0.665
1346	10	1.1915	0.9756	0.3966	0.5480	0.593	0.562
1347	24	1.8551	2.0539	0.2127	1.8960	0.896	0.923
2102	29	0.2214	0.2584	0.2542	0.0129	0.016	0.050
2104	35	0.8883	0.6824	0.3114	0.3657	0.544	0.536
2105	63	0.1671	0.3793	0.2664	0.1162	0.298	0.306
3410	34	-0.5655	0.4581	0.3758	0.0901	0.180	0.197
3411	62	0.0091	0.3669	0.4185	-0.0493	-0.141	-0.134
3412	2	0.3279	0.4529	0.7982	-0.3453	-0.763	-0.763
3616	20	1.0390	0.6350	0.1016	0.5361	0.840	0.844
3622	20	1.3560	1.3560	0.1168	0.9442	0.888	0.901

Table 8 (Continued)

Line	n	$\sum_{i=1} \Delta_i / n$	s_I^2	s_{III}^2	s_{IV}	r_{II}(19a)	r_{II}(19b)
3625	39	0.0862	0.3721	0.2488	0.1172	0.331	0.315
3707	11	0.2534	0.5518	0.3316	0.2551	0.399	0.498
3705	21	0.6974	0.6819	0.3717	0.3435	0.455	0.504
3706	29	-0.1343	0.3123	0.2761	0.0348	0.116	0.111
1348	30	-0.2680	0.5443	0.2656	0.2230	0.512	0.410
1349	13	1.1304	1.1356	0.4327	0.6779	0.619	0.597
1350	19	0.7340	0.6066	0.3845	0.1672	0.366	0.276
1351	4	0.4287	0.2995	0.0937	0.1769	0.687	0.591
1352	15	-0.2156	0.5101	0.4275	0.0410	0.162	0.080
1353	30	-0.6970	0.5688	0.2372	0.3246	0.583	0.571
1354	6	0.6809	0.9311	1.0203	-0.0178	-0.096	-0.019
1355	24	1.1674	1.0747	0.2625	0.6754	0.756	0.628
1356	36	0.6912	1.3318	0.3676	0.9180	0.724	0.689
1357	35	-0.4747	0.5853	0.3861	0.1994	0.340	0.341
1358	35	-0.3634	0.3405	0.2010	0.1485	0.410	0.436
1359	12	-0.2025	0.3531	0.4039	-0.0382	-0.144	-0.108
1360	20	0.4278	0.1878	0.0802	0.1156	0.573	0.616
1361	28	-0.7732	0.5238	0.1802	0.3621	0.565	0.691
1362	13	0.6092	0.9157	0.4202	0.3770	0.541	0.412
1363	24	0.1133	0.2876	0.3357	-0.0669	-0.167	-0.233
1364	30	-0.3798	0.5329	0.3847	0.1624	0.278	0.305
1365	2	-0.1853	1.0085	1.9827	-0.9742	-0.366	-0.966
1366	17	-0.1377	0.4357	0.2728	0.0901	0.374	0.207
1367	34	-0.6166	0.3875	0.2528	0.1393	0.347	0.360
1368	8	0.7612	0.4067	0.1921	0.2335	0.528	0.574
1369	12	1.1002	0.8667	0.3092	0.6251	0.643	0.721
1370	13	0.0336	0.3888	0.2645	0.1097	0.320	.282
1371	13	-1.2304	.8735	.1052	.8098	.880	.927
1372	39	0.2214	.4971	.1451	.3540	.708	.712
1373	31	0.0732	.4862	.2948	.2017	.394	.415
Pooled results:		0.2816	0.6627	0.3414		0.422	0.415

$\Sigma(n-1)=1419$

Number of lines: 62

This formula yields an estimate of twice the systematic error (a) as defined in chapter 6, i.e. the one-ways difference:

$$2\hat{a} = \sum_{i=1}^{n} \Delta_i / n$$

All 62 lines give the following mean value for $\hat{a}$:

$$\hat{a} = 0.141 \text{ mm}/\sqrt{\text{km}}$$

with the standard error

$$s_{\hat{a}} = 0.064 \text{ mm}/\sqrt{\text{km}}$$

as

$$\hat{a}/s_{\hat{a}} = 2.20 > t(61) = 1.67$$

We conclude that there is a significant systematic difference (bias) between forward and backward levelling. The confidence interval for $\hat{a}$ becomes (0.034, 0.247) mm/$\sqrt{\text{km}}$. (α = 5%).

8. Solution for Systemetic Errors and Correlated Random Errors

From formulas (13), (15), (17) and (18) we obtain the following independent equations

$$s_p^2 = \hat{a}^2 k + s^2(1+2r_{II}) \qquad (23a)$$

$$s_w^2 = \tfrac{1}{2}s^2(1+2r_{II})\,(1+r_I) \qquad (23b)$$

$$s_I^2 = 2\hat{a}^2 + s^2\,(1-r_I) \qquad (23c)$$

$$s_{III}^2 = s^2\,(1-r_I)\,(1-r_{II}) \qquad (23d)$$

where

$$k = \frac{1}{21}\sum_{i=1}^{21} m_i^2/n_i = 37.566$$

and $\hat{a}$, s^2, r_I, r_{II} are unknowns. The solution of (23) becomes

$$\hat{a}^2 = \frac{2\,(s_p^2 - s_w^2 - s_I^2 + s_{III}^2) - s_I^2}{2\,(k-3)}$$

$$r_I = \frac{2\,s_w^2}{s_p^2 - \hat{a}^2 k} - 1$$

$$r_{II} = 1 - \frac{s_{III}^2}{s_I^2 - 2\hat{a}^2}$$

$$s^2 = \frac{s_I^2 - 2\hat{a}^2}{1 - r_I}$$

Inserting $s_p^2 = 4.32$, $s_w^2 = 0.640$, $s_I^2 = 0.6627$ and $s_{III}^2 = 0.3414$ mm^2/km

$\hat{a}^2 = 0.08758$ $\quad$ $\hat{a} = 0.295$ mm/$\sqrt{km}$

$r_I = 0.243$

$r_{II} = 0.300$

$s^2 = 0.644$ $\quad$ $s = 0.802$ mm/$\sqrt{km}$

Comparing with section. 7.2 we notice that $\hat{a}=0.295$ mm/$\sqrt{km}$ is outside the confidence interval for $\hat{a}$. Somehow the above variances are not compatible with the systematic error detected by analysing the levelling lines. We may then ask whether there are any values for $\hat{a}$ within the confidence interval of section 7.2 consistent with the mean square errors (ariances). Next we try to answer this question.

Let us assume that $\hat{a}$ is a priori given. Then we may solve for the three remaning unknowns by (23 b-d). The solution is

$$r_{II} = 1 - \frac{s_{III}^2}{s_I^2 - 2\hat{a}^2} \tag{24a}$$

$$r_I = \frac{2s_w^2\,(1-r_{II}) - s_{III}^2\,(1+2r_{II})}{2s_w^2\,(1-r_{II}) + s_{III}^2\,(1+2r_{II})} \tag{24b}$$

and

$$s^2 = \frac{s_I^2 - 2\hat{a}^2}{1 - r_I} \qquad (24c)$$

Another estimate of s^2 is obtained from (23a)

$$s^2 = \frac{s_p^2 - \hat{a}^2 k}{1 + 2r_{II}} \qquad (24d)$$

If we assume that the two estimates are independet we can use their ratio to test wether they are identical. This is the case if their ratio (max/min) is inferier to $F(18,18)=2.22$. (α = 5%), where F is taken from an F- distribution table [18 degrees of freedom = 21 (loops) minus 3 (unknowns)] . The upper confidence limit for $\hat{a}$ is 0,247 mm/$\sqrt{km}$ (see section 7.2). Inserting this value (and $s_w^2=0.6400$, $s_I^2=0.627$, $s_{III}^2=0.3414$. $s_I^2=4.32$) into (24a-b) we get

$r_{II}= 0.369$ $\qquad$ $r_I= 0.154$

and from (24c) and (24d)

$s^2= 0.639$ $\qquad$ $s^2= 1.167$

Thus we obtain the test parameter

$$T=\frac{1.167}{0.639}=1.82 < F\ (18,18).$$

and subsequently $\hat{a}=0.247$ is compatible with the variances. In Table 9 we indclude several estimates of r_{II}, r_I, and s^2 as computed by formules (24 a-b).

Table 9. r_{II}, r_I and s^2 computed by parametes (24 a-d) for various $\hat{a}$.
F=2.22. Risk level: 5%. The estimetes are consistant with the error model for T<F.

$\hat{a}$	r_{II}	r_I	s^2 (c)	s^2 (d)	T ratio	Consistent with error model
0.247	0.369	0.154	0.639	1.167	1.82	Yes
0.220	0.397	0.116	0.640	1.395	2.18	Yes
0.215	0.401	0.109	0.640	1.434	2.24	No
0.210	0.406	0.103	0.641	1.47	2.29	No
0.200	0.414	0.092	0.614	1.54	2.40	No

From the table we arrive at the following confidence intervals

$$0.22 < a < 0.25 \quad mm/\sqrt{km}$$
$$0.37 < \rho_{II} < 0.40$$
$$0.12 < \rho_{I} < 0.15$$

and finally by pooling estimates from (24 c) and (24 d)

$$0.95 < \sigma < 1.01 \quad mm/\sqrt{km}$$

These are the most probable limits of the estimated parameters in agreement with obtained variances (s_D^2, s_W^2, s_I^2, and s_{III}^2) and the confidence interval for $\hat{a}$ derived in section 7.2.

9. Concluding Remarks

The investigation has proved that no systematic errors occur in the two-ways levelling. Hence the RMS closing error 0.8 mm/$\sqrt{km}$ should be regarded as the noise level of the method. It is interesting that the systematic refraction error occuring in classical levelling (see Remmer, 1975 and 1980) has not been detected in the motorized technique. This error source is obviously reduced due to the increase of the line of sight above the ground. On the other hand, the analysis isbased merely on the data obtained for sections. Systematic errors, hidden in this analysis, might prove to be significant when including data from each setup.

The standard error difference between $\Delta h/2$ and w indicate that there are systematic errors in the one-way observations and/or that the random errors are correlated. Possible systematic errors are obviously eliminated in the two-ways means. A proper discrimination between a systematic and a random error source could be obtained by a special experiment: repeted levelling (forward and backward) of one loop. Then the random errors would vanish in the means. In any case a strong negative correlation (-0.7) is found between the one-way levellings.

In section 7.2 we have shown that there is a significant bias (2a) between forward and backward levelling. The 5% confidence interval of the bias is [0.068 0.494] mm/$\sqrt{km}$. In chapter 8 we assume that the systematic error is of the same magnitude (a) but with opposite sign for forward and backward levelling. The error model, introduced in chapter 6, is solved for a systematic error (in accordance with section 7.2), correlation coefficients between neighbouring sections (ρ_{II}) and between forward and backward levelling (ρ_I) and finally for variance of unit weight (σ^2). The resulting confidence intervals are

$$0.22 < a < 0.25 \text{ mm}/\sqrt{\text{km}}$$

$$0.95 < \sigma < 1.01 \text{ mm}/\sqrt{\text{km}}$$

$$0.12 < \rho_I < 0.15$$

$$0.37 < \rho_{II} < 0.40$$

The study has shown that the overall RMS closing error reduces from 2.1 mm/$\sqrt{\text{km}}$ to 0.8 mm/$\sqrt{\text{km}}$ when turning from one-way techniques to two-ways mean levelling. From this result and from the point of view of the low possibility of detecting gross errors at an early stage, we do not recommend the application of the motorized technique in one-way levelling.

Finally we emphasize that most of the analysis and tests were based upon the assumption of independent levelling polygons. We assume that this shortcoming is negligible and does not justify a strict but much more laborious investigation.

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